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Abstract

In this note, we show a global weak existence result for a two dimensionnal Compressible Primitive Equations for atmosphere dynamics modeling.

Keywords: Atmosphere modelling, *a priori* estimates, Existence theorem.

1. Introduction

The Primitive Equations (PEs) of the atmosphere modeling are fundamental equations of geophysical fluid mechanics (see e.g. [5]). In the hierarchy of models for geophysical flow, the PEs are situated between the so-called non hydrostatic models and shallow water models. They are generally derived from the full set of geophysical fluid equations. Owing to the difference of the depth and length scale, the derivation consists to replace the momentum conservation equation for the vertical velocity by the hydrostatic equation, in the same spirit of the derivation of the shallow water equations ([2, 4]). We refer to [3] for the mathematical formulation and existence results for these equations. We investigate a simple version of the Compressible Primitive Equations (CPEs) for atmosphere dynamics where we do not deal with complexe phenomena as solar heating effects or the amount of water in the air (as done in [6]). This model is already introduced in [1], and they obtain a global existence theorem for weak solutions for a model, called model problem, close to the one presented in this note. As a straightforward consequence of the existence result [1], we prove the global solvability of the initial boundary value problem for the simplified CPEs.

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2. A global existence result for the simplified CPEs

We start with the simplified version of CPEs which, in cartesian coordinates, is written:

$$\left\{\begin{array}{lcl} \partial_t\rho + \partial_x(\rho u) + \partial_y(\rho v) & = & 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_y(\rho u v) + \partial_x p & = & \partial_x(\nu_1(\rho)\partial_x u) + \partial_y(\nu_2(\rho)\partial_y u), \\ \partial_y p & = & -\rho g \end{array}\right. \tag{1}$$

where x, y stand for the horizontal space and vertical variable. ρ is the density, $\mathbf{u} = (u, v)$ is the velocity of the fluid with u (resp. v) the horizontal (resp. vertical) component, p is the pressure given by the equation of state:

$$p(\rho) = c^2\rho \text{ for some given constant } c, \tag{2}$$

The constant c is $c^2 = RT$ where R is the specific gas constant for the air and T the temperature, assumed constant. The turbulence viscosities are (ν_1, ν_2) for the horizontal and vertical direction and is written:

$$\begin{array}{l} \nu_1(t,x,y) = \nu_0 e^{-g/c^2 y} \text{ for some given constant } \nu_0, \\ \nu_2 \text{ any given function.} \end{array} \tag{3}$$

We assume that the motion of the medium occurs in a rectangular domain

$$\Omega = \{(x,y);\, 0 < x < l,\, 0 < y < h\}$$

and we prescribe the following boundary conditions as:

$$\begin{array}{l} u_{|x=0} = u_{|x=l} = 0, \\ v_{|y=0} = v_{|y=h} = 0, \\ \partial_y u_{|y=0} = \partial_y u_{|y=h} = 0 \end{array} \tag{4}$$

and the initial conditions as:

$$\begin{array}{l} u_{|t=0} = u_0(x,y), \\ \rho_{|t=0} = \xi_0(x) e^{-g/c^2 y} \end{array} \tag{5}$$

where ξ_0 is assumed to be a bounded strictly positive function:

$$0 < m \leqslant \xi_0 \leqslant M < \infty.$$

Then, we state the main result:

Theorem 1. *Suppose that initial data (ξ_0, u_0) have the properties:*

$$(\xi_0, u_0) \in W_2^1(\Omega), \quad u_{0|x=0} = u_{0|x=l} = 0.$$

Then $\rho(t,x,y)$ is a bounded strictly positive function and (1)-(4) has a weak solution in the following sense: a weak solution of (1)-(4) is a collection (ρ, u, v) of functions such that $\rho \geqslant 0$ and

$$\rho \in L^\infty(0,T; W_2^1(\Omega)), \quad \partial_t\rho \in L^2(0,T; L^2(\Omega)), \tag{6}$$

$$u \in L^2(0,T;W_2^2(\Omega)) \cap W_2^1(0,T;L^2(\Omega)), \quad v \in L^2(0,T;L^2(\Omega)) \tag{7}$$

which satisfies (1) in the distribution sense; in particular, the integral identity holds for all $\phi_{|t=T} = 0$ *with compact support:*

$$\begin{aligned} &\int_0^T \int_\Omega \rho u \partial_t \phi + \rho u^2 \partial_x \phi + \rho u v \partial_z \phi + \rho \partial_x \phi \, dx + \rho v \phi dz dt \\ &= - \int_0^T \int_\Omega u \Delta \phi \, dx dz dt + \int_\Omega u_0 \xi_0 \phi_{|t=0} \, dx dz \end{aligned} \tag{8}$$

Proof of Theorem 1: For simplicity, we assume that $l = h = 1$, $g = c^2$, $\nu_1(t,x,y) = e^{-y}$ and $\nu_2(t,x,y) = e^y$. Then using the hydrostatic approximation $\partial_y p = -\rho$, the pressure p can be written as a tensorial product as follows;

$$p(\rho(t,x,y)) = \rho(t,x,y) = \xi(t,x) e^{-y} \tag{9}$$

where ξ is an unknown function. Then, Equation (9) provides a stratified structure to the density ρ. Indeed, assume ξ known, then the structure of ρ is given as a function of y variable only. Thus, this fact suggests to use the following change of variables:

$$z = 1 - e^{-y}. \tag{10}$$

As we will see, this change of variables (10) allows to write the initial model (1) of unknowns $(\rho(t,x,y), u(t,x,y), v(t,x,y))$ as a model more simple with unknowns $(\xi(t,x,y), u(t,x,y), w(t,x,y))$ where w defines the vertical velocity in the new coordinates as

$$w(t,x,z) = e^{-y} v(t,x,y). \tag{11}$$

Multiplying Equations (1) by e^y and using the change of variables (10) provides the model, called model problem by the authors [1]:

$$\left\{ \begin{array}{lcl} \partial_t \xi + \partial_x(\xi u) + \partial_z(\xi w) & = & 0 \\ \partial_t(\xi u) + \partial_x(\xi u^2) + \partial_z(\xi u w) + \partial_x \xi & = & \partial_x(\partial_x u) + \partial_z(\partial_z u) \\ \partial_z \xi & = & 0 \end{array} \right. \tag{12}$$

where w is defined as (11). This is exactly the model studied by *Gatapov et al* [1], derived from Equations (1) by neglecting some terms, in which they provide the following global existence result:

Theorem 2 (B. Gatapov and A.V. Kazhikhov 2005). *Suppose that initial data* (ξ_0, u_0) *have the properties:*

$$(\xi_0, u_0) \in W_2^1(\Omega), \quad u_{0|x=0} = u_{0|x=1} = 0.$$

Then $\xi(t,x)$ *is a bounded strictly positive function and (12) has a weak solution in the following sense: a weak solution of (12) satisfying the boundary conditions*

$$\begin{aligned} &u_{|x=0} = u_{|x=1} = 0, \\ &w_{|y=0} = w_{|y=1} = 0, \\ &\partial_z u_{|z=0} = \partial_z u_{|z=1} = 0, \end{aligned}$$

is a collection (ξ, u, w) of functions such that $\xi \geqslant 0$ and

$$\xi \in L^\infty(0,T;W_2^1(0,1)), \quad \partial_t \xi \in L^2(0,T;L^2(0,1)), \tag{13}$$

$$u \in L^2(0,T;W_2^2(\Omega)) \cap W_2^1(0,T;L^2(\Omega)), \quad w \in L^2(0,T;L^2(\Omega)) \tag{14}$$

which statisfy (12) in the distribution sense; in particular, the integral identity holds for all $\phi_{|t=T} = 0$ with compact support:

$$\begin{aligned}&\int_0^T \int_\Omega \xi u \partial_t \phi + \xi u^2 \partial_x \phi + \xi u w \partial_z \phi + \xi \partial_x \phi \, dxdzdt \\ &= -\int_0^T \int_\Omega u \Delta \phi \, dxdzdt + \int_\Omega u_0 \xi_0 \phi_{|t=0} \, dxdz\end{aligned} \tag{15}$$

Now, assume that initial data (ξ_0, u_0) have the properties:

$$(\xi_0, u_0) \in W_2^1(\Omega), \quad u_{0|x=0} = u_{0|x=1} = 0,$$

then $\xi(t,x)$ is a bounded strictly positive function and there exist (ξ, u, w) such as:

$$\xi \in L^\infty(0,T;W_2^1(0,1)), \quad \partial_t \xi \in L^2(0,T;L^2(0,1)),$$
$$u \in L^2(0,T;W_2^2(\Omega)) \cap W_2^1(0,T;L^2(\Omega)), \quad w \in L^2(0,T;L^2(\Omega))$$

which statisfy (12) in the distribution sense. Moreover, by a simple change of variables $z = 1 - e^{-y}$ in integrals, we have the following properties:

- $||\rho||_{L^2(\Omega)} = \alpha ||\xi||_{L^2(\Omega)}$,
- $||\nabla_x \rho||_{L^2(\Omega)} = \alpha ||\nabla_x \xi||_{L^2(\Omega)}$,
- $||\partial_y \rho||_{L^2(\Omega)} = \alpha ||\xi||_{L^2(\Omega)}$

where

$$\alpha = \int_0^{1-e^{-1}} (1-z)\, dz.$$

We deduce then,

$$||\rho||_{W_2^1(\Omega)} = \alpha W_2^1(\Omega)$$

which provides

$$\rho \in L^\infty(0,T;W_2^1(\Omega))$$

and

$$\partial_t \rho \in L^2(0,T;L^2(\Omega)).$$

Estimates on u remains true. Again, by a simple change of variables in integrals, the fact that $v \in L^2(0,T;L^2(\Omega))$ is obtained from the inequality:

$$\begin{aligned}||v||_{L^2(\Omega)} &= \int_0^1 \int_0^1 |v(t,x,y)|^2 \, dy \, dx = \int_0^1 \int_0^1 |e^y w(t,x,y)|^2 \, dy \, dx \\ &= \int_0^1 \int_0^{1-e^{-1}} \left(\frac{1}{1-z}\right)^3 |w(t,x,z)|^2 \, dz \, dx \\ &< \left(\frac{1}{1-h}\right)^3 ||w||_{L^2(\Omega)}.\end{aligned}$$

■

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